The formula for propagation of error given in Chapter 7 is useful for calculating variance for some mappings of one or more random variables (x,y,…) to a third random variable (z). However, there are important cases when a more general approach is needed. These cases arise when the probability density function $pdf_x(x)$ of one random variable (x) is known, and you want to calculate $pdf_y(y)$ given the mapping function y=f(x). If you can determine $pdf_y(y)$, then the mean and variance of y as well as higher moments can be readily determined.

The basic scheme for calculating a $pdf_y(y)$ in terms of a functionally related $pdf_x(x)$ is that the probability{y1≤y≤y2} = probability{x1≤x≤x2}, where y1=f(x1) and y2=f(x2) which leads to the following

$$\int_{y1}^{y2} pdf_y(y)\,dy = \int_{x1}^{x2} pdf_x(x)\,dx \qquad \text{Eq. 1}$$

Note the use of subscripts for the pdf's to indicate which is for x and which is for y. Also recall that the integral over the full range of each pdf() is unity. Importantly, Eq. 1 assumes that the mapping function y=f(x) is monotonically increasing so that a one-to-one mapping of x to y occurs. This holds for simple linear mapping functions such as y=ax+b. However, and for an important case such as $y=x^2$, Eq. 1 can be modified as follows

$$\int_{y1}^{y2} pdf(y)\,dy = \int_{x1}^{x2} pdf(x)\,dx + \int_{-x1}^{-x2} pdf(x)\,dx \qquad \text{Eq. 2}$$

since the two ranges of integration for x (one for x>0 and the other for x<0) map to the same range of integration for y. Since both integrals are areas under pdf's and both have the same area this leads to a simpler equation for this case.

$$\int_{y1}^{y2} pdf(y)\,dy = 2\int_{x1}^{x2} pdf(x)\,dx \qquad \text{Eq. 2a}$$

Higher order polynomials and periodic functions also follow this multi integration range scheme though we will not deal with those here.

If we look at the limiting case for Eq. 1 where the ranges of integrations are reduced to dy and dx then we see that

$$\begin{aligned} pdf_y(y)\,dy &= pdf_x(x)\,dx \\ pdf_y(y) &= pdf_x(x)\frac{dx}{dy} \end{aligned} \qquad \text{Eq. 3}$$

Rearranging Eq. 3 and using the absolute value of the derivative leads to

$$pdf_y(y) = \frac{pdf_x(x)}{\left|\frac{dy}{dx}\right|} \quad \text{Eq. 4}$$

The absolute value is needed since dy may be increasing or decreasing as dx increases but the areas associated with the pdf's is always positive. We want everything on the right side of Eq. 4 to be in terms of y and we can accomplish this by expressing x as $f^{-1}(y)$

$$pdf_y(y) = \frac{pdf_x(f^{-1}(y))}{\left|\frac{dy}{dx}\right|} = \frac{pdf_x(x(y))}{\left|\frac{dy}{dx}\right|} \quad \text{Eq. 5}$$

This assumes that $f^{-1}(y)$ exists, which will be true for pdf's we will study. Eq. 5 shows that $pdf_y(y)$ can be calculated by replacing the argument of $pdf_x(x)$ with $x=f^{-1}(y)$ and dividing by the absolute value of dy/dx. Therefore $pdf_x()$ serves as the template for the $pdf_y()$ which is scaled by 1/|dy/dx| to calculate $pdf_y()$.

For students interested in how this technique can be expanded to deal with more complex functional relationships look for chapters on "Probability and Functions" in classical statistics texts such as the one by Papoulis.

Example 1. Given that x is a zero mean Gaussian random variable and y=ax+b, what is $pdf_y(y)$?

Given: $$pdf_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x}{\sigma_x}\right)^2}$$

Calculate: $f^{-1}(y) = x(y) = (y-b)/a$

$|dy/dx| = |a|$

Using Eq. 5 we see that $pdf_y(y)$ is

$$pdf_y(y) = \frac{1}{\sqrt{2\pi}\,|a|\,\sigma_x} e^{-\frac{1}{2}\left(\frac{y-b}{a\sigma_x}\right)^2}$$

We see from this result that $pdf_y(y)$ is also Gaussian. There are two changes worthy of noting, 1) the location of the peak and mean value (μ_y) is y=b and 2) the standard deviation (σ_y) is equal to $|a|\sigma_x$. Note. Shifting x by "b" shifts the mean location of the Gaussian distribution to b, and multiplying x by "a" multiplies the standard deviation by |a|.

A classical use of this simple linear mapping is to transform a Gaussian random variable with known mean and standard deviation to the normal (Gaussian) distribution with $\mu = 0$ and $\sigma = 1$. Transforming a Gaussian distributed random value (x) to the normal distribution (z) is done using the mapping function $z=(x-\mu_x)/\sigma_x$ where μ_x and σ_x are the mean and standard deviation for the random variable x and z is the normally distributed random value with $\mu = 0$ and $\sigma = 1$. Probability tables are available in most statistical texts for this normal distribution, so probabilities for x between any range can be calculated by determining the appropriate range of z in a normal distribution table.

An inverse transform $x_i = \sigma_x z_i + \mu_x$ can be used to generate random variables with mean = μ_x and standard deviation = σ_x using a normal random number generator to create the z_i. Normal random number generators can be found in various software packages (Mathcad, Mathlab, Mathematica, etc.).

Example 2. Given that x is a zero mean Gaussian random variable and $y=ax^2$, with $a>0$, what is $pdf_y(y)$? (Note that $y \geq 0$)

Given: $$pdf_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x}{\sigma_x}\right)^2}$$

Calculate: $$f^{-1}(y) = x(y) = \pm\sqrt{\frac{y}{a}}$$

$$|dy/dx| = |2ax| = \left|2\sqrt{ay}\right|$$

Solving for $pdf_y(y)$ makes use of Eq. 2a since there are two values of x for each value of y. Using Eq. 5 with this modification this leads to

$$pdf_y(y) = \frac{1}{\sqrt{2\pi a y}\sigma_x} e^{-\frac{1}{2}\frac{y}{a\sigma_x^2}}$$

Note that $pdf_y(y)$ is no longer a Gaussian distribution, as should have been expected, since the mapping function was nonlinear and particularly since $y>0$ though x can be positive or negative. The new pdf has the form of a Gamma distribution function.

Probability Density Function for Noise in MRI.

The above background was provided in part to help understand the nature of the noise distribution function in magnetic resonance imaging. In most cases a pixel value in an MR image is calculated as follows

$$S_m = ((S_r + x)^2 + (S_i + y)^2)^{1/2} \qquad \text{Eq. 6}$$

where S_m is the signal magnitude and S_r and S_i are raw signals acquired in the real and imaginary channels of the RF receiver. The magnitude signal is calculated for each pixel within the MR image, so Eq. 6 is taken as one such pixel value. Here x and y are assumed to be zero mean random noise (Gaussian random variables) with the identical standard deviations (σ). Upon expansion of the inner terms in Eq. 6 we get

$$S_m = (S_r^2 + S_i^2 + 2S_rx + 2S_iy + x^2 + y^2)^{1/2}. \qquad \text{Eq. 7}$$

The first two terms in this equation are the theoretical signals arising from the real and imaginary channels. These are the desired signals and if the SNR is sufficiently large $S_m \sim (S_r^2 + S_i^2)^{1/2}$. This case it is not always possible, especially for functional MRI (fMRI) studies.

As we saw when we calculated the $pdf_y(y)$ for $y=ax^2$, **the pdf of the signal magnitude $pdf_m(S_m)$ is not Gaussian**. In fact, in areas where the signal is nonzero $pdf_m(S_m)$ is quite complex. However, <u>for cases where the signal terms can be assumed to be zero</u> (e.g. outside the body) only the last two terms in Eq.7 are non-zero and we find that $pdf_m()$ follows a Raleigh distribution with the following form

$$pdf_m(\rho) = \frac{\rho}{\sigma^2} e^{-\frac{1}{2}\left(\frac{\rho}{\sigma}\right)^2} \qquad \text{Eq. 8}$$

where $\rho = (x^2 + y^2)^{1/2}$ is the noise magnitude expressed in 2-D polar coordinates (ρ,θ=0). Using $pdf_m(\rho)$ from Eq. 8 we can find the following relationships:

mean of noise magnitude = $\mu_\rho = \sigma\sqrt{\frac{\pi}{2}} \approx 1.25\sigma$

standard deviation of noise magnitude = $\left(2 - \frac{\pi}{2}\right)^{1/2} \sigma \approx 0.655\sigma$

If we measure the mean value of a region of an MR image where the signal = 0, it should be ~ 1.25σ where σ is the standard deviation of ρ. We can use this property to estimate σ. A similar relationship is seen if we measure the standard deviation of the magnitude signal.

However, there is a problem when attempting to calculate the SNR from the signal magnitude inside the body, since as seen in Eq. 7 we cannot easily subtract off the mixed signal and noise terms to accurately estimate the signal. One approach to deal with this problem and to estimate SNR is based on analysis of an image of a cylindrical phantom. A large region of interest (ROI) can be placed over the phantom portion of the image and the mean value of this ROI used to estimate the signal. In this case the ROI mean value is a good approximation of the true signal since the ROI should contain several hundred pixels, and the average of the mixed signal-noise terms in Eq. 7 should therefore average toward zero. The noise can be evaluated using an ROI outside the phantom. **Caution, do not include noise due to phase artifacts**. The mean from this signal-less ROI can be used to calculate the noise as described above. Using the signal estimated from the mean of the signal magnitude within the phantom and the standard deviation (σ) estimated from the mean of the signal-less region an SNR can be estimated.

Tissue-to-tissue SNR

The above method is good for specifying SNR of an object relative to a zero signal, but the SNR for low-contrast signals is the focus of most signal detection problems in imaging, and this low-contrast SNR is based on the signal difference between adjacent tissues. For practical purposes, we can draw ROIs in adjacent tissues to estimate the tissue-to-tissue SNR. This SNR is calculated using the mean (μ) and standard deviation (σ) from each tissue ROI as follows:

$$SNR_{tissue1-tissue2} = \frac{\mu_{tissue1} - \mu_{tissue2}}{(\sigma_{tissue1} + \sigma_{tissue2})/2} \qquad \text{Eq. 9}$$

The ROIs should be approximately the same size. Eq. 9 is consistent with the definition of SNR in Chapter 1 and takes into account the inhomogeneity of the tissues in its estimation noise.

Noise Bias in the Magnitude Signal.

An important issue arises when attempting to estimate T1 or T2 from relaxation data at numerous time points along a signal magnitude relaxation curve $S_m(t)$. This is particularly important for T2 measurements where we want to sample early and late time points in the relaxation curve and may be modeling the relaxation as a multi-exponential process. From Eq. 7 we see that noise that is added to the signal due to the x^2 and y^2 terms is always positive, and this leads to a positive bias relative to the true signals, i.e. the measured signal magnitude will tend to be larger than the true signal magnitude. Interestingly, if we chose to fit the square of Eq. 7 (sometimes called the power) we can correct for this bias by subtracting μ_ρ^2 from each time sample power value, $S_m^2(t)$. This removes this bias since it is the expected value of $\rho^2 = (x^2 + y^2)$ at each time sample power value. Other errors (i.e. those due to the cross terms in Eq. 7) are left to be minimized by the fitting process.