

Scaling and shifting of functions.

An integrable function $f(x)$ can be height scaled, width scaled, and shifted along x .

$$a \cdot f\left(\frac{x-b}{c}\right) \begin{cases} a = \text{height scale} \\ c = \text{width scale} \\ b = \text{origin shift} \\ \int_{-\infty}^{+\infty} a \cdot f\left(\frac{x-b}{c}\right) dx = a \cdot c \int_{-\infty}^{+\infty} f(x) dx \end{cases}$$

The integral of the height and width scaled function is the integral of the original function multiplied by the product of the height and width scales $a \cdot c$. We refer to “ $a \cdot c$ ” as the area scale, and if $a=1/c$ then the area is unchanged by the scaling. The shift term does not affect area as long as the limits of integration span the range of the function.

Scaling and shifting of template functions in Chapter 5.

The normalized sinc function is

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \begin{cases} = 1 & @ x = 0 \\ = 0 & @ x = \pm n \text{ for } n > 0 \text{ and } n \text{ is an integer} \\ \int_{-\infty}^{+\infty} \text{sinc}(x) dx = 1 \end{cases}$$

The general form of the sinc function may be scaled and shifted

$$a \cdot \text{sinc}\left(\frac{x-b}{c}\right) = a \cdot \frac{\sin\left[\pi\left(\frac{x-b}{c}\right)\right]}{\pi\left(\frac{x-b}{c}\right)} \begin{cases} = a & @ x = b \\ = 0 & @ \frac{x-b}{c} = \pm n \text{ for } n > 0 \text{ and } n \text{ is integer} \\ \int_{-\infty}^{+\infty} a \cdot \text{sinc}\left(\frac{x-b}{c}\right) dx = a \cdot c \int_{-\infty}^{+\infty} \text{sinc}(x) dx = a \cdot c \end{cases}$$

NOTE: If $a = 1/c$ then the area for the scaled sinc function remains unity.

The scaling and shifting properties for other normalized template functions (rectangle, triangle, and Gaussian) follow the trend given for the sinc function.

Relationship between rect and sinc functions.

Several interesting results are seen when we evaluate the Fourier transform of $f(x) = \text{rect}(x/a)$.

$$\begin{aligned}
1) \quad \mathfrak{F}[f(x)] &= \mathfrak{F}\left[\text{rect}\left(\frac{x}{a}\right)\right] = \int_{-\frac{a}{2}}^{+\frac{a}{2}} e^{-2\pi i u x} dx \\
2) \quad &= \frac{e^{-2\pi i u \frac{a}{2}} - e^{2\pi i u \frac{a}{2}}}{-2\pi i} = \left(\frac{e^{\pi i u a} - e^{-\pi i u a}}{2i}\right) \frac{1}{\pi i} = \frac{\sin(\pi u a)}{\pi i} = a \frac{\sin(\pi u a)}{\pi u a} \\
3) \quad \text{so } \mathfrak{F}[f(x)] &= F(u) = a \text{ sinc}(au)
\end{aligned}$$

From this simple exercise we see that $f(x) = \text{rect}(x/a)$ with width scale = a transforms to $F(u) = a \text{ sinc}(au)$ with height scale = a and width scale of $1/a$.

Result 1. Since $f(0) = 1$ the integral of $F(u)$ should be unity. This is easily confirmed since the integral of $\text{sinc}(u)$ is unity and the area scale is also unity for $a \cdot \text{sinc}(au)$.

Result 2. Since $F(0) = a$, the integral of $f(x) = a$. This is easily confirmed for $\text{rect}(x/a)$ since its height=1 and its width = a .

Result 3. Setting $g(x) = 1/a \text{ rect}(x/a)$ leads to $G(u) = \text{sinc}(au)$ and area reverses Results 1 & 2.

Result 4. $a = 1$ leads to the classic unit area templates, $f(x) = \text{rect}(x)$ and $F(u) = \text{sinc}(u)$, as Fourier transform pairs.

Result 5. For the spatial domain function $\text{rect}(x/a)$ the corresponding frequency domain $a \cdot \text{sinc}(au)$ first goes to zero at $u=1/a$. This supports a sinc function that goes to zero at non integer multiples of π . This zero point is independent of height scaling.

Result 6. In the limit as “ a ” goes to infinity $\text{rect}(x/a)$ approaches a unity valued constant over all x . We know that the Fourier transform of such a constant is $\delta(u)$ leading to the following:

$$\begin{aligned}
\delta(u) &\Rightarrow \mathfrak{F}\left[\text{rect}\left(\frac{x}{a}\right)\right] = \int_{-\frac{a}{2}}^{+\frac{a}{2}} e^{-2\pi i u x} dx \\
\text{limit } a &\Rightarrow \infty
\end{aligned}$$