

The autocorrelation function is a measure of similarity between a data set and a shifted copy of the data as a function of shift magnitude. Correlation analysis is used to find periodic patterns in noisy data, characterize similarity patterns in data compression, and measurement of spatial resolution of an image receptor with uniform white noise as the input. For medical imaging a major use of autocorrelation is for the measurement of film and screen spatial resolution.

The definition of the autocorrelation function is similar to that for the autocovariance function. The autocorrelation function is defined as $E\{I(x) I(x+\Delta)\}$ where $E\{\}$ is the expectation operation and the autocovariance function defined as $E\{[I(x)-\langle I(x)\rangle][I(x+\Delta)-\langle I(x)\rangle]\}$. If $I(x)$ is a zero-mean function, then $\langle I(x)\rangle = 0$ and the two definitions give the same result. In fact, for most medical imaging applications $I(x)$ is transformed to a zero mean function before the correlation function is calculated so we will consider autocorrelation function = autocovariance function.

The autocorrelation function will be designated as $C_x(\Delta)$ where

$$C_x(\Delta) = E\{I(x) I(x+\Delta)\}. \tag{Eq 1}$$

Example 1. A simple periodic function.

Following is an example of how to calculate the autocorrelation function for a simple set of data that repeats every three samples. First lets take a look a $C_x(\Delta)$ with $\Delta = 0$

| | |
|---------------|---|
| $I(x)$ | ...+1, -1, 0, +1, -1, 0, +1, -1, 0, ... |
| $I(x+0)$ | ...+1, -1, 0, +1, -1, 0, +1, -1, 0, ... |
| $I(x) I(x+0)$ | ...+1, +1, 0, +1, +1, 0, +1, +1, 0, ... |

$$C_x(0) = \frac{1}{N} \sum_{x=1}^N I(x)I(x+0) = \frac{1}{9} \sum_{x=1}^9 I(x)I(x+0) = \frac{6}{9} = \frac{2}{3} \tag{Eq 2}$$

The average was taken over three periods ($N = 9$) in this case, but in general the extent will be much larger. Also, note that as the numerator increases by 2 for each period the denominator increases by 3 such that $C_x(0) = 2/3$ when averaged over whole periods. For a large number of samples (i.e., for $N = 1000$), averaging over an exact number of periods in not as important. With the period of 3 999 samples will cover 333 periods. However, 1000 samples will cover for 333 periods with 1 sample only from the 334th period so the average will not be over full periods only. This would be much worse if you had 10 samples covering 3 1/3 periods. If the period is known then calculation range should span multiples of the period.

Now continue and calculate $C_x(\Delta)$ with $\Delta = 1$

| | | | | | | | | | |
|----------|-----|---------|---------|--------|---------|---------|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $I(x)$ | ... | +1, -1, | 0, +1, | -1, 0, | +1, -1, | 0, ... | | | |
| $I(x+1)$ | ... | -1, 0, | +1, -1, | 0, +1, | -1, 0, | +1, ... | | | |

$I(x) I(x+1) \quad \dots -1, 0, 0, -1, 0, 0, -1, 0, 0, \dots$

$$C_x(1) = \frac{1}{N} \sum_{x=1}^N I(x)I(x+1) = \frac{1}{9} \sum_{x=1}^9 I(x)I(x+1) = \frac{-3}{9} = \frac{-1}{3} \quad (\text{Eq 3})$$

Note that as the numerator in this calculation increases by -1 for each period the denominator increases by 3 such that $C_x(1) = -1/3$. If the calculations are continued for various spacing you will get the following

$$\begin{aligned} \Delta &= \dots -3, -2, -1, 0, +1, +2, +3 \dots \\ C_x(\Delta) &= \dots 2/3, -1/3, -1/3, 2/3, -1/3, -1/3, 2/3 \dots \end{aligned} \quad (\text{Eq. 4})$$

Inspection of this autocorrelation function leads to the following characteristics:

- The autocorrelation function is positive (2/3) at $\Delta = 0$. Further it can be shown that the value of $C_x(0) \geq C_x(\Delta)$ for all Δ . For non-repeating functions $C_x(0) > C_x(\Delta)$, i.e. $C_x(\Delta)$ will be a maximum at $\Delta = 0$. This will clearly be the case when we investigate detector spatial resolution using random noise.
- The autocorrelation function in this example repeats with a period = 3, the period of $I(x)$.
- The autocorrelation function is $-1/3$ for $\Delta = \pm 1, \pm 2$, is of opposite sense and 1/2 the magnitude of that a $\Delta = 0$.

Since $I(x)$ is a zero mean function in this example, autocorrelation (or autocovariance) with $\Delta = 0$ is the variance of $I(x)$, or stated symbolically mathematically $C_x(0) = \sigma^2$. With this identity in mind we can interpret $C_x(\Delta)$ as the covariance between a zero-mean function $I(x)$ and its shifted version $I(x+\Delta)$. A normalized measure of autocorrelation, normalized relative to the variance, is $R_x(\Delta) = C_x(\Delta)/C_x(0)$. Therefore $R_x(0) = 1$ and $R_x(\Delta) \leq 1$ for all other values of Δ providing a standardized measure of similarity as a function of displacement.

Example 2. Autocorrelation function of sinusoids.

Since cosine is a periodic function we must integrate over integer number of periods to ensure a proper calculation of its autocorrelation function. The basic equation is

$$C(\Delta) = \frac{1}{nT} \int_{-\frac{nT}{2}}^{\frac{nT}{2}} \cos\left(\frac{2\pi}{T}x\right) \cos\left[\left(\frac{2\pi}{T}\right)(x + \Delta)\right] dx \quad (\text{Eq. 5})$$

From trigonometry $\cos A \cos B = 1/2 \cos(A-B) + 1/2 \cos(A+B)$ and setting $A = (2\pi/T)x$ and $B = (2\pi/T)x + (2\pi/T)\Delta$ and using $\cos(\theta) = \cos(-\theta)$ we get

$$C(\Delta) = \frac{1}{nT} \int_{-\frac{nT}{2}}^{\frac{nT}{2}} \frac{1}{2} \cos\left(\frac{2\pi}{T}\Delta\right) dx + \frac{1}{nT} \int_{-\frac{nT}{2}}^{\frac{nT}{2}} \frac{1}{2} \cos\left[\left(\frac{2\pi}{T}\right)(2x + \Delta)\right] dx \quad (\text{Eq. 6})$$

$$C(\Delta) = \frac{1}{2} \cos\left(\frac{2\pi}{T}\Delta\right) \quad (\text{Eq. 7})$$

since the second integral is zero when integration range is a multiple of the period (T).

An identical answer is seen for the sine function. Note that for this example $C(0) = 1/2$ and as for the previous example $C(\Delta)$ has the same period (T) as the original sinusoid.

In general the following are true for all autocorrelation functions for $I(x)$:

- $C(0) = \sigma^2$ at $\Delta=0$ the autocorrelation function is variance of $I(x)$
- $C(-\Delta) = C(\Delta)$ autocorrelation functions are symmetric
- $C(0) \geq C(\Delta)$ maximum @ zero displacement ($\Delta=0$)
- $C(\Delta) = \sigma^2(\Delta)$ autocorrelation function = variance as function of Δ
- $R(\Delta) = C(\Delta)/C(0)$ normalized autocorrelation function

Autocorrelation and Autoconvolution. There is a natural similarity between the autocorrelation function and convolving a function with itself (autoconvolution). Autoconvolution is as follows

$$f \otimes f = \int f(y)f(x - y)dy \tag{Eq. 8}$$

whereas for autocorrelation there is no reflection in the second term leading to

$$C(x) = \int f(y)f(y - x)dy = \int f(y)f(y + x)dy \tag{Eq. 9}$$

where ‘y’ is a dummy ‘x’ variable for integration and $f(x)$ is a real (i.e. not complex) function. The x in these equations can be replaced by Δ since both measure distance from $x=0$. Graphing the two forms in Eq. 9 will reveal why they are identical on integration.

Power Spectral Density. The Fourier transform of $C(\Delta)$ also has a simple form based on the Fourier transform of $f(x)$ which is $F(u)$

$$\varphi(u) = \mathfrak{F}\{C(\Delta)\} = F(u)F^*(u) = |F(u)|^2 \tag{Eq. 10}$$

$\varphi(u)$ is called the **power spectral density (PSD)** of $f(x)$. The power spectral density and the autocorrelation function form a Fourier transform pair with the following example in 2-D:

$$\varphi(u, v) = \iint C(\Delta_x, \Delta_y)e^{-2\pi i(u\Delta_x + v\Delta_y)} d\Delta_x d\Delta_y \tag{Eq. 11}$$

and

$$C(\Delta_x, \Delta_y) = \iint \varphi(u, v)e^{2\pi i(u\Delta_x + v\Delta_y)} dudv \tag{Eq. 12}$$

In general the following are true for the power spectral density function:

- $C(0,0) = \sigma^2 = \iint \varphi(u,v)dudv$ Integral of PSD = variance $f(x)$

- $\varphi(u,v)$
- $\varphi(u,v) \geq 0$
- $\varphi(u,v) = \varphi(-u,-v)$

PSD is real (Eq. 10)
 PSD is non-negative (Eq. 10)
 PSD is symmetric

A summary of autocorrelation and power spectral density functions for various input functions is helpful in understanding their use:

| Table 1. General Form of Autocorrelation and Power Spectral Density Functions. | | |
|---|-------------------------------|--------------------------------|
| input function | C(Δ) | $\varphi(u)$ |
| sinusoid ($\mu=0$) | cosine | delta functions |
| Gaussian | Gaussian | Gaussian |
| delta function | delta function | constant |
| random noise ($\mu=0$) | delta function | constant |

The power spectral density of zero mean random noise is called the **Wiener Spectrum** and is usually written as $W(u)$ rather than $\varphi(u)$ to make this explicit. The autocorrelation function $C(\Delta)$ and power spectrum $W(u)$ properties of random noise can be exploited to

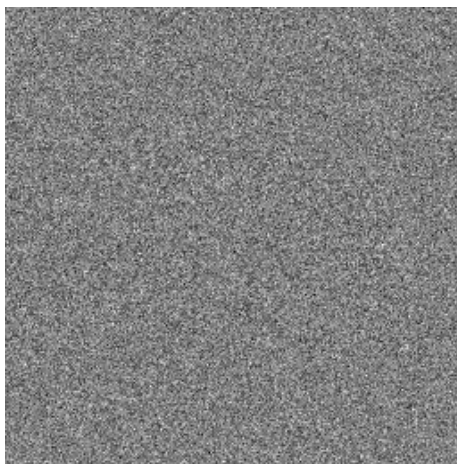


Figure 1. 2-D image $[i(x,y)]$ of random values ($\mu = 0, \sigma = 1$).

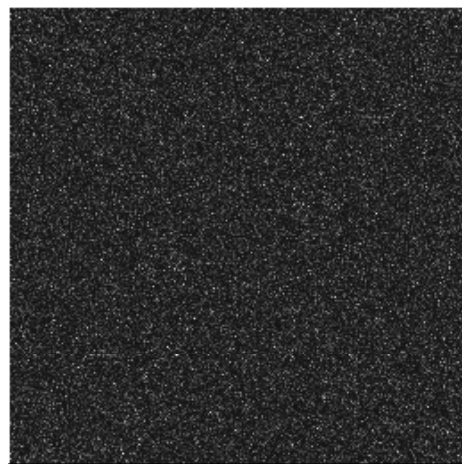


Figure 2. $|I(u,v)|^2$ is the power or Wiener spectrum of $i(x,y)$

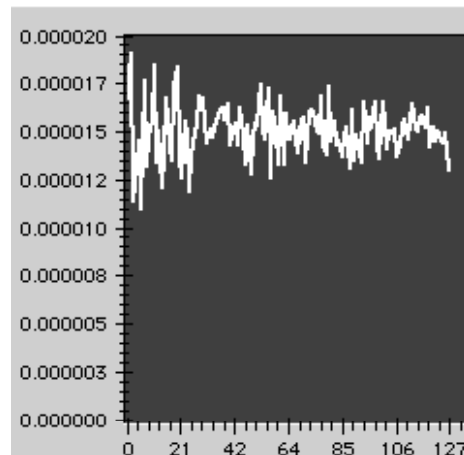


Figure 3. Radial frequency plot $W(\rho)$ of Wiener spectrum from Figure 2.

determine spatial resolution of film/screen systems. Figure 1 is a random noise 256x256 image $i(x,y)$ where the mean =0 and the standard deviation = 1. Figure 2 is its Wiener spectrum $\{ |I(u,v)|^2 \}$ where $|I(u,v)|$ is the magnitude of the Fourier transform of $i(x,y)$. The origin $u=0, v=0$ is at the center of Figure 2.

The integral of $|I(u,v)|^2$ is equal to 1 as predicted since it should be equal to $\sigma^2 = 1$ for the image $i(x,y)$. Figure 3 is a graph of the Wiener spectrum $[W(\rho)]$ expressed as a function of the distance from the origin ρ (i.e., the radial frequency). Note that the highest frequency is ~ 128 cycles or line pairs as expected for a 256x256 image. Note also that for this graph the data appears to vary about a mean response. Since every location in W should vary randomly about its mean and the sum of all locations is the integral of $|I(u,v)|^2 = 1$ the mean value can be estimated by dividing unity by matrix size 256^2 ($1/256^2 = 1.53 \times 10^{-5}$).

If the Wiener spectra in Figure 3 is fitted with a straight line it will have slope = 0 and magnitude equal to the above mean value. The constant magnitude across all possible frequencies demonstrates that the Wiener spectrum of random noise, if unmodified by the system transfer function, is made up of equal amplitudes at all frequencies. This noise is often called “white” due to the fact that white light also has a uniform mix of light of over a broad range of frequencies or wavelengths. Figures 1-3 illustrate the calculation of the Wiener spectrum in an ideal imaging system, i.e. one with a constant magnitude system transfer function $|H(\rho)| = k$. The Wiener spectra is therefore just the square of the system transfer function $|H(\rho)|$ or $W(\rho) = |H(\rho)|^2$ for this ideal system. This relationship is can be used to test the resolution capabilities of an imaging system when the input is “white” noise.

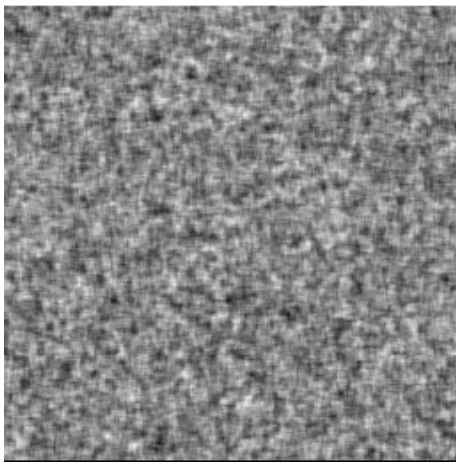


Figure 4. Random noise image of system modeled with a 9x9 Gaussian point spread function.

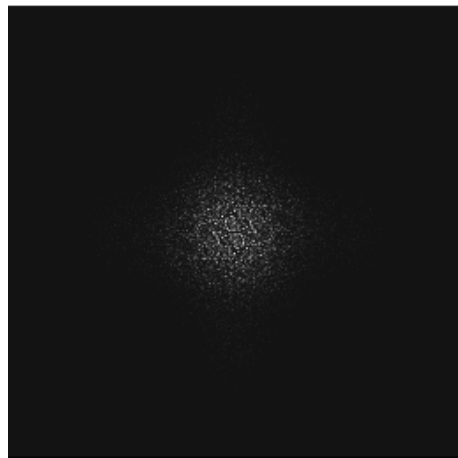


Figure 5. Wiener spectra of system modeled with a 9x9 Gaussian point spread function.

Figures 4-6 provide some insight into how the Wiener spectrum relates to system resolution. Let’s assume that an imaging system alters the 2-D random image $i(x,y)$ of

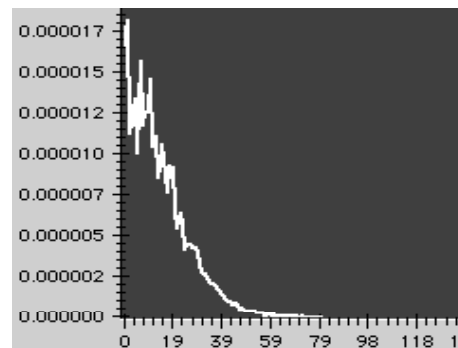


Figure 1 due to blurring as a result of point spread function. Figure 4 models this using a 9x9 Gaussian PSF applied to $i(x,y)$ to simulate the imaging system blurring. The blurring is modeled in the frequency domain as $I_s(u,v) = I(u,v)H(u,v)$, where $I(u,v)$ is the Fourier transform of $i(x,y)$ and $H(u,v)$ is the simulated system transfer function.

The 2-D Wiener spectrum of the blurred noise image in Figure 4 is given in Figure 5 and as a radial plot $[W_s(\rho)]$ in Figure 6. It follows that $W_s(\rho) = |H_s(\rho)|^2$ or $|H_s(\rho)| = [W_s(\rho)]^{1/2}$. This latter equation states that the square root of the Wiener spectrum for a system is the magnitude of its system transfer function. Therefore $|H_s(\rho)|$ is similar to MTF. The system MTF is estimated as $[W_s(\rho)/W_s(0)]^{1/2}$. This division sets $MTF(0)=1$.

While the Wiener spectrum can be calculated without using the autocorrelation function it is instructive to analyze system spatial blurring by direct analysis of the autocorrelation function of noise. The rationale is that the autocorrelation function for random noise input to an ideal imaging system simulates a perfect point spread function for testing [i.e., expected response = $\delta(x,y)$]. Broadening of this expected point response is due to imaging system blurring, i.e. the $PSF(x,y)$ of the system is not a delta function. We say that the system PSF introduces short-range correlations, and this is reflected as a broadening of the autocorrelation function of random noise. In the spatial domain this is modeled as

$$C_s(\Delta x, \Delta y) = C(\Delta x, \Delta y) \otimes [PSF(-x,-y) \otimes PSF(x,y)] \tag{Eq. 13}$$

where the broadened autocorrelation function $C_s(\Delta x, \Delta y)$ is different from the ideal autocorrelation function due to convolution with $PSF(x,y)$ twice. An example of this is provided in Figures 7 & 8.

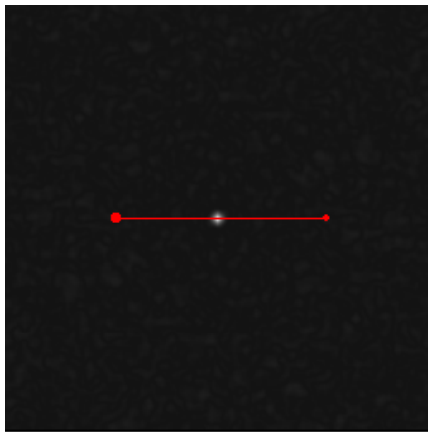


Figure 7. Autocorrelation of $i(x,y)$ after smoothing with a 9x9 Gaussian filter.

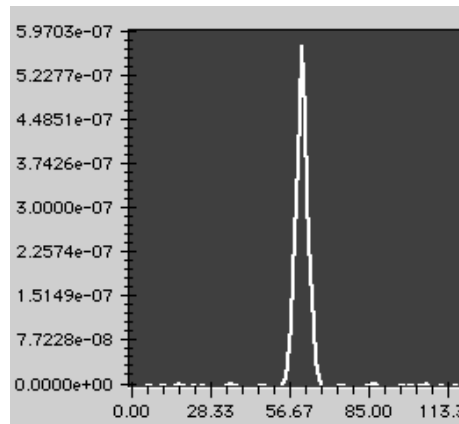


Figure 8. Profile representing PSF^2 from Figure 7.

The similarity of the autocorrelation analysis of a random noise image and the direct application of smoothing is seen from the similarity of Figures 7,8 and 9,10. Taking the square root of the profile data in Figures 8 & 10 provides an estimate of the system PSF.

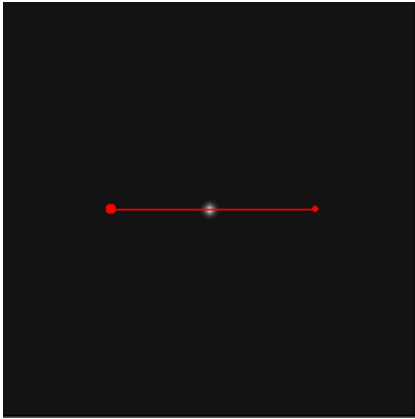


Figure 9. Point that was smoothed twice with a 9x9 Gaussian filter.

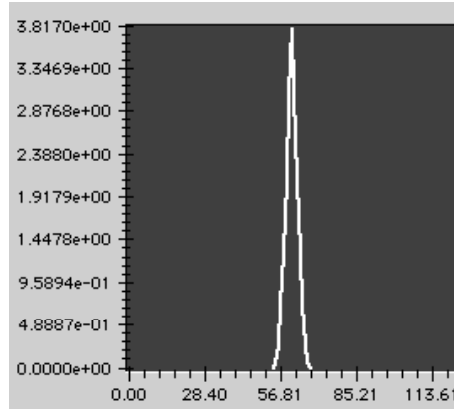


Figure 10. Profile along PSF in Figure 9.

Measuring Film/Screen Resolution. The autocorrelation function and Wiener spectra are both sensitive to low frequency background variations and this limits their use in Nuclear Medicine, where such changes in uniformity are nearly always present. However, both can be used to assess spatial resolution of film/screen systems without confounding effects of other components of spatial resolution. A uniform random image can be made by exposing a small region of a film/screen at a large distance from the x-ray tube. This image is free of effects of focal spot size, grid, or magnification. The film is then scanned with a small aperture microdensitometer and the recorded film density converted to relative exposure using calibration data for the film/screen combination. The film/Screen MTF is calculated as the square root of the normalized $W(\rho)$ for this data. MTF data is then corrected for the drop-off due to the scanner aperture.

The uniform noise presented to the film/screen system is quantum noise (i.e. white noise spectrally). The added noise in the developed image is also due to random processes in both the screen and the film, and since these are assumed to be independent the random noise for these are additive. However, blurring by film and screen alter the MTF, which leads to an overall MTF with decreasing response as frequency increases (Fig. 11).

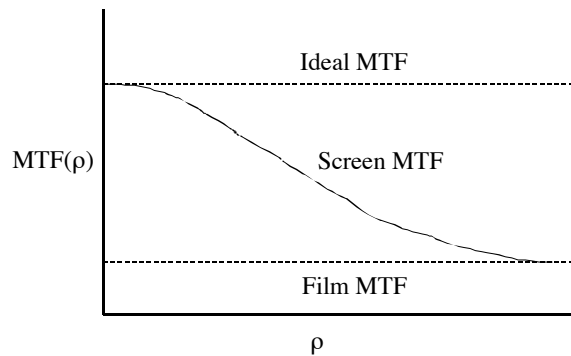


Figure 11. MTF from film/screen system.

Since the film MTF is approximately constant over the range when the Screen MTF is falling off, these can be literally separated.

More on DQE.

The system equation for DQE (8-2) presented in Chapter 8 does not account for the natural loss in signal contrast with increasing frequency of imaging systems. Since mean random noise levels at the input and output of an imaging system are approximately constant with increasing frequency, the change in signal contrast or modulation with frequency leads to changes in SNR(f). A more general equation for DQE is therefore

$$DQE(f) = \frac{SNR_{out}^2(f)}{SNR_{in}^2(f)} \quad (\text{Eq. 14})$$

Using logic that was applied for the development of the Rose model equation $SNR_{in}(f)$ can be expressed as a function of $C_{in}(f)$ and N_{in} , the number of photons from an area of interest within the background

$$SNR_{in}^2(f) = C_{in}^2(f)N_{in} = MTF_{in}^2(f)N_{in} = N_{in} \quad (\text{Eq. 15})$$

For sinusoidal response $MTF_{in}^2(f)$ is equivalent to $C_{in}^2(f)$. Since we can assume an ideal frequency response at the input of the system $MTF_{in}^2(f) = 1$ for all frequencies, simplifying SNR_{in} . We can calculate $SNR_{out}(f)$ for a sinusoid using $signal_{out}(f) = C_{out}(f)$ times the mean signal out as follows

$$SNR_{out}^2(f) = \frac{C_{out}^2(f)\bar{s}_{out}^2}{\sigma_{out}^2} = MTF_{out}^2(f)\left(\frac{\bar{s}}{\sigma}\right)_{out}^2 \quad (\text{Eq. 16})$$

Again for sinusoidal response $MTF_{out}^2(f)$ is equivalent to $C_{out}^2(f)$. $MTF_{out}(f)$ is determined by the overall imaging system frequency response, i.e. due to focal spot size, magnification, grids, scatter, film, screen, etc. The average signal out (\bar{s}_{out}) and its standard deviation (σ_{out}) are calculated using the system's output units. The ratio of Eq.16 to Eq.15 gives the system DQE(f)

$$DQE(f) = \frac{\left(\frac{\bar{s}}{\sigma}\right)_{out}^2}{N_{in}} MTF_{out}^2(f) \quad (\text{Eq. 17})$$

where

$$DQE(0) = \frac{\left(\frac{\bar{s}}{\sigma}\right)_{out}^2}{N_{in}} \quad (\text{Eq. 18})$$

since $MTF_{out}^2(0) = 1$. Note that the mean and standard deviation in Eq.18 are just those that would be measured if the signal were uniform, and $DQE(0)$ is the same as the baseline DQE we determined using Eq.8-2 in Chapter 8. Since $MTF_{out}^2(f) \leq 1$, **DQE(f) diminishes with increasing frequency indicating less effective use of the input quanta.**

DQE(f) for subcomponents of a system can be calculated using appropriately determined $\text{SNR}_{\text{in}}(f)$ and $\text{SNR}_{\text{out}}(f)$ for the component, a case where $\text{SNR}_{\text{in}}(f)$ is no longer a constant.

In most applications MTF(f) is calculated from the point spread function. It may be necessary to correct for sample spacing and aperture size used when digitizing the output image to obtain an accurate value for $\text{MTF}_{\text{out}}(f)$, i.e. if these aren't sufficiently small.